Invariance principle for extension of hydrodynamics: Nonlinear viscosity

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Invariance of nonhydrodynamic variables is put forward as a working principle of extending hydrodynamics into a highly nonequilibrium domain. Following this principle, the leading modification of the viscosity due to the gradient of the average velocity is derived explicitly from nonlinear moment Grad equations [Commun. Pure Appl. Math. **2**, 331 (1949)]. [S1063-651X(97)01802-3]

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The problem of extending the hydrodynamics into a highly nonequilibrium domain beyond the Navier-Stokes approximation remains one of the central and only partially resolved problems of the kinetic theory. The classical Chapman-Enskog (CE) method [1] provides, in principle, an opportunity of such extension, and leads to the solution in the form of a series in powers of the Knudsen number ϵ, the ratio between the mean-free path of a particle, and the scale of variations of the hydrodynamic quantities. The CE solution results in a formal expansion of the stress tensor and of the heat flux vector in the balance equations for the density, the momentum, and the energy. The first-order term (ϵ) in the latter expansions leads to the Navier-Stokes equations, while further terms are known as the Burnett (ϵ^2) and the super-Burnett (ϵ^3) corrections [1].

However, as is now well known, even in the simplest linear case, the Burnett and the super-Burnett hydrodynamics violate the stability of the equilibrium for perturbations with sufficiently short wavelength [2]. The possible root of this violation is the poor convergency of the Taylor-like expansions underlying the CE method. The situation is even more complicated in the nonlinear domain, where the very structure of the higher-order terms of the CE expansion remains largely unknown.

In this paper we consider the problem of extension of the hydrodynamics into a highly nonequilibrium domain on the basis of a simpler (in comparison to the Boltzmann equation) model—nonlinear ten-moment Grad equation [3]. Even in this model, however, the CE expansion appears to be rather complicated in the full setting. Therefore we address another, more restrictive problem: what is the *leading* correction to the Navier-Stokes approximation when the characteristic value of the average velocity is comparable to the heat velocity? It will be possible to indicate the terms of the CE expansion which are relevant to this leading correction (namely, the terms containing the highest powers of the gradient of flux). Further, however, it will be necessary to *sum up all the relevant terms*. At this point, instead of using the CE expansion, we will take another route, namely, the method of invariant manifold [4], which leads to the result more directly, and which does not require the summation to be done explicitly. The correction amounts to a flux-dependent nonlinear viscosity in the Navier-Stokes equations, and results are presented in an explicit form.

Throughout the paper, we use the system of units where Boltzmann's constant and the particle's mass are equal to 1, and ρ, u, and T are the density, the average velocity, and the temperature, respectively. The pressure p is $p=\rho T$, according to the system of units adopted. The starting point is the set of one-dimensional nonlinear Grad equations [3] for the hydrodynamic variables ρ, u, and T, coupled to the nonhydrodynamic variable σ, where σ is the xx component of the stress tensor:

$$\partial_t \rho = -\partial_x(\rho u), \tag{1a}$$

$$\partial_t u = -u\partial_x u - \rho^{-1}\partial_x p - \rho^{-1}\partial_x \sigma, \tag{1b}$$

$$\partial_t T = -u\partial_x T - (2/3)T\partial_x u - (2/3)\rho^{-1}\sigma\partial_x u, \tag{1c}$$

$$\partial_t \sigma = -u\partial_x \sigma - (4/3)p\partial_x u - (7/3)\sigma\partial_x u - \frac{p}{\mu(T)}\sigma. \tag{1d}$$

Here $\mu(T)$ is the temperature-dependent viscosity coefficient. We will adopt the form $\mu(T)=\alpha T^{\gamma}$, which is characteristic to the point-center models of the particle's collisions, where γ varies from $\gamma=1$ (Maxwellian molecules) to $\gamma=1/2$ (hard spheres), and where α is a dimensional factor [1].

Equation (1) provides a relatively simple (in comparison to the Boltzmann equation) model of a coupling of the hydrodynamic variables, ρ, u, and T, to the nonhydrodynamic variable σ, and corresponds to a heat nonconductive case. The goal is to shorten the description, and to get a closed set of equations with respect to the variables ρ, u, and T. That is, we have to express the nonhydrodynamic variable σ in terms of the hydrodynamic variables ρ, u, and T in Eqs. (1b) and (1c). After that, the three equations (1a)–(1c) will provide the desired closed system of hydrodynamic equations.

If the CE method is applied for this purpose to the system (1), then one introduces the formal parameter ϵ^{-1} in front of the term $p\mu^{-1}\sigma$ in Eq. (1d), and the solution is sought in the form of a series, $\sigma_{\mathrm{CE}}=\Sigma_0^{\infty}\epsilon^{n+1}\sigma^{(n)}$, where $\sigma^{(0)}$ gives the Navier-Stokes closure relation, and where the further terms $\sigma^{(n)}$ are due to a well-known recurrence procedure [1]. However, even in the framework of the system (1), the algebraic structure of this recurrence procedure is fairly involved. Moreover, looking ahead, we will have to deal with the

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terms of the CE expansion to arbitrary order n. Therefore a different but equivalent (in the context of the problem) route of constructing the closure relations [4] will be used.

First, we will introduce this route of computations considering a simplified case of Eq. (1), namely, the linear form of this system. Using dimensionless variables $\delta p=\Delta p/p_0$, $\delta u=\Delta u/\sqrt{T_0}$, $\delta\sigma=\Delta\sigma/p_0$, where 0 labels the equilibrium, and Δ labels deviations of the corresponding quantities ($u_0=0$ and $\sigma_0=0$), introducing the scaled time-space variables $t'=[p_0/\mu(T_0)]t$, and $x'=[p_0/(\sqrt{T_0}\mu(T_0)]x$, linearizing Eq. (1), and after Fourier transforming in space and omitting primes, we come to the following system for the Fourier variables δu_k, δp_k, and $\delta\sigma_k$:

$$\partial_t\delta p_k=-(5/3)ik\delta u_k, \tag{2a}$$

$$\partial_t\delta u_k=-ik\delta p_k-ik\delta\sigma_k, \tag{2b}$$

$$\partial_t\delta\sigma_k=-(4/3)ik\delta u_k-\delta\sigma_k. \tag{2c}$$

The closure relation $\delta\sigma_k(\delta u_k,\delta p_k,k)$ for Eq. (2) was found exactly in the framework of the CE method [5]. A different route [4] will be used now. We seek the solution $\delta\sigma_k(\delta u_k,\delta p_k,k)$ in the following form, prompted by the linearity of the problem:

$$\delta\sigma_k=ikA(k)\delta u_k-k^2B(k)\delta p_k, \tag{3}$$

where the k-dependent functions A and B are to be determined.

The invariance principle [4] requires that a closure relation should be form invariant under both the microscopic and the macroscopic dynamics. In the context of the system (2), the ''microscopic'' dynamics is defined by Eq. (2c) (the equation for the nonhydrodynamic variable $\delta\sigma_k$), while the ''macroscopic'' dynamics is due to Eqs. (2a) and (2b) (the equations for the hydrodynamic variables δp_k and δu_k). Thus, in accord with [4], the closure relation (3) should be form invariant under both the dynamics due to Eq. (2c), and the dynamics due to Eqs. (2a) and (2b). The ''microscopic'' time derivative of the function σ_k is obtained when the expression (3) is substituted into the right-hand side of Eq. (2c):

$$\partial_t^{\text{micro}}\delta\sigma_k=-(4/3)ik\delta u_k-ikA\delta u_k+k^2B\delta p_k. \tag{4}$$

On the other hand, the ''macroscopic'' time derivative of the closure relation (3) due to Eqs. (2a) and (2b) reads

$$\begin{aligned}\partial_t^{\text{macro}}\delta\sigma_k&=\frac{\partial\delta\sigma_k}{\partial\delta u_k}\partial_t\delta u_k+\frac{\partial\delta\sigma_k}{\partial\delta p_k}\partial_t\delta p_k\\&=ik(\tfrac{5}{3}B+A^2)k^2\delta u_k+k^2A(1-k^2B)\delta p_k.\end{aligned} \tag{5}$$

The requirement of equivalence of both the derivatives, $\partial_t^{\text{micro}}\delta\sigma_k=\partial_t^{\text{macro}}\delta\sigma_k$, for each δu_k and δp_k results in the following system of equations for the functions A and B:

$$-\tfrac{4}{3}-A=k^2(\tfrac{5}{3}B+A^2),\quad B=A(1-k^2B). \tag{6}$$

The *invariance equation* (6) concludes essentially the problem of constructing the closure relation (3). Now we are able to make the following identification: the invariance equation (6) is *equivalent* to the system of equations for the functions A and B, as resulting from the exact summation of the CE expansion [5]. Notice also that an explicit use of the Knudsen number ϵ was avoided.

With the identification just made, let us turn back to the nonlinear case (1). Our goal is to compute the correction to the Navier-Stokes closure relation, $\sigma_{\text{NS}}=-(4/3)\mu\partial_x u$, for high values of the average velocity. Consider first the Burnett correction as derived by the CE method from Eq. (1):

$$\begin{aligned}\sigma_B=&-\frac{4}{3}\mu\partial_x u+\frac{8(2-\gamma)}{9}\mu^2p^{-1}(\partial_x u)^2\\&-\frac{4}{3}\mu^2p^{-1}\partial_x(\rho^{-1}\partial_x p).\end{aligned} \tag{7}$$

The correction of the desired type is given by the nonlinear term proportional to $(\partial_x u)^2$. Each further nth term of the CE expansion contributes, among others, the nonlinear term proportional to $(\partial_x u)^{n+1}$. Such terms can be named *high-speed* terms since they dominate the rest of the contributions in each order of the CE expansion when the characteristic average velocity is comparable to the heat velocity [6]. Simple dimensional analysis leads to a conclusion that such terms are of the form $\mu g^n\partial_x u$, where $g=p^{-1}\mu\partial_x u$ is dimensionless. Therefore the CE expansion for the function σ may be formally rewritten as

$$\begin{aligned}\sigma_{\text{CE}}=&-\mu\left\{\frac{4}{3}-\frac{8(2-\gamma)}{9}g+r_2g^2+\cdots+r_ng^n+\cdots\right\}\\&\times\partial_x u+\cdots.\end{aligned} \tag{8}$$

The series in the brackets is the collection of the high-speed contributions of interest, coming from *all* the orders of the CE expansion, while the dots outside the brackets stand for the terms of other nature. Thus the high-speed correction to the Navier-Stokes closure relation in the framework of the Grad equations (1) takes the form

$$\sigma_{nl}=-\mu R(g)\partial_x u, \tag{9}$$

where $R(g)$ is the function represented by a formal subsequence of CE terms in the expansion (8). The function R can be viewed also as a dynamic modification of the viscosity μ due to the gradient of the average velocity.

We will now turn to the problem of an explicit derivation of the function R, Eq. (9). Following the principle of invariance as explained above, we first compute the microscopic derivative of the function σ_{nl} by substituting the expression (9) into the right-hand side of Eq. (1d):

$$\begin{aligned}\partial_t^{\text{micro}}\sigma_{nl}=&-u\partial_x\sigma_{nl}-(4/3)p\partial_x u\\&-(7/3)\sigma_{nl}\partial_x u-[p/\mu(T)]\sigma_{nl}\\=&\{-\tfrac{4}{3}+\tfrac{7}{3}gR+R\}p\partial_x u+\cdots,\end{aligned} \tag{10}$$

where dots denote the terms irrelevant to the closure relation (9) [such terms appear, of course, because Eq. (9) is not the exact closure relation].

Second, computing the macroscopic derivative of the closure relation (9) due to Eqs. (1a)–(1c), we have

$$\partial_t^{\text{macro}}\sigma_{nl} = -[\partial_t\mu(T)]R\partial_x u - \mu(T)\frac{dR}{dg}[\partial_t g]\partial_x u - \mu(T)R\partial_x[\partial_t u]. \quad (11)$$

In the latter expression, the time derivatives of the hydrodynamic variables should be replaced with the right-hand sides of Eqs. (1a)–(1c), where, in turn, the function σ should be replaced by the function σ_{nl}, Eq. (9). After some computation, we come to the following:

$$\partial_t^{\text{macro}}\sigma_{nl} = \left\{ gR + \frac{2}{3}(1-gR)\left(\gamma gR + (\gamma-1)g^2\frac{dR}{dg}\right)\right\} \times p\partial_x u + \cdots. \quad (12)$$

Here dots stand again for the terms irrelevant to the present analysis.

Equating the relevant terms in Eqs. (10) and (12), we come to the invariance equation for the function R:

$$(1-\gamma)g^2(1-gR)\frac{dR}{dg} + \gamma g^2R^2 + [\tfrac{3}{2} + g(2-\gamma)]R - 2 = 0. \quad (13)$$

Equation (13) plays the same role in the problem under consideration as Eq. (6) plays in the linear case, and is our main result.

For Maxwellian molecules ($\gamma=1$), Eq. (13) simplifies considerably, and is algebraic:

$$g^2R^2 + (\tfrac{3}{2}+g)R - 2 = 0. \quad (14)$$

The solution which recovers the Navier-Stokes closure relation in the limit of small g reads

$$R_{MM} = \frac{-3-2g+3\sqrt{1+(4/3)g+4g^2}}{4g^2}. \quad (15)$$

The function R_{MM}, Eq. (15), is given in Fig. 1. Notice that the function R_{MM} is positive for all values of its argument g, as is appropriate for the viscosity factor, while the Burnett approximation to the function R_{MM} violates the positivity.

For other models ($\gamma\neq 1$), the invariance equation (13) is a rather complicated nonlinear ordinary differential equation (ODE) with the initial condition $R(0)=4/3$ (the Navier-Stokes condition). Several ways to derive analytic results are possible. One possibility is to expand the function R into powers of g, in the point $g=0$. This way brings us back to the original subseries of the CE expansion [see Eq. (8)]. Instead, we take the opportunity offered by the parameter γ. Introduce another parameter $\beta=1-\gamma$, and consider an expansion:

$$R(\beta,g) = R_0(g) + \beta R_1(g) + \beta^2 R_2(g) + \cdots.$$

Substituting this expansion into the invariance equation (13), we derive $R_0(g)=R_{MM}(g)$,

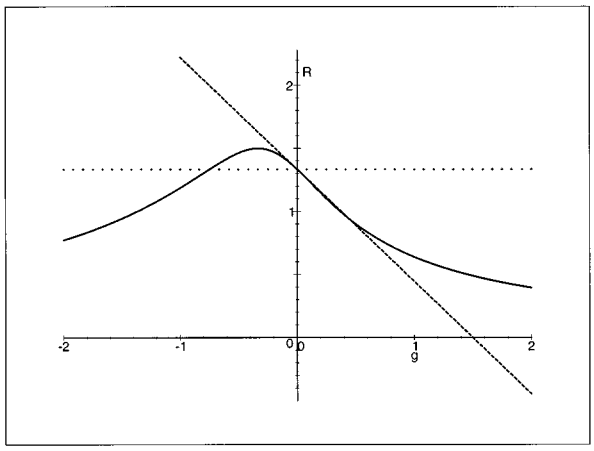

FIG. 1. Viscosity factor $R(g)$ for Maxwellian molecules: exact (solid), the Burnett approximation (dash), the Navier-Stokes approximation (dots).

$$R_1(g) = -\frac{(1-gR_0)gR_0 + (3/2)g^2(dR_0/dg)}{2g^2R_0 + g + (3/2)}, \quad (16)$$

etc. That is, the solution for other models is constructed in the form of a series with the exact solution for the Maxwellian molecules as the leading term. For hard spheres ($\beta=1/2$), the result to the first-order term reads $R_{\text{HS}}\approx R_{MM} + (1/2)R_1$, and is given in Fig. 2. The features of the approximation obtained are qualitatively the same as in the case of the Maxwellian molecules. The question of convergency of the procedure remains, however, beyond this paper, and we conclude with a discussion.

(i) The main feature of the above example of extending the hydrodynamic description into a highly nonequilibrium and nonlinear domain can be expressed as follows: this is an *exact partial summation* of the CE expansion. ''Partial'' means that the relevant high-speed terms, dominating the other contributions in the limit of the high average velocity, were accounted to all the orders of the original CE expansion. ''Exact'' means that, though we have used the formally different route, the result is indeed the sum of the relevant

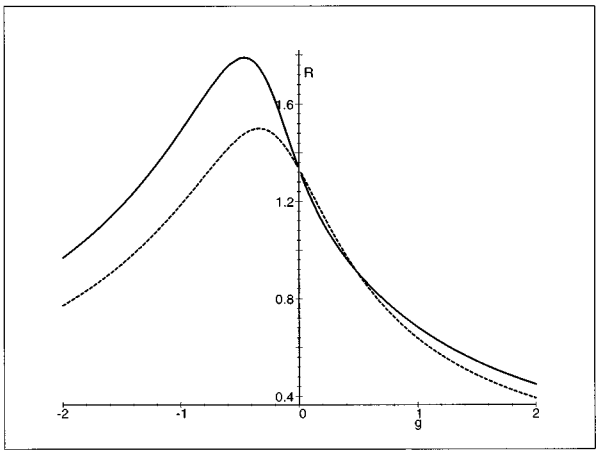

FIG. 2. Viscosity factor $R(g)$ for hard spheres: the first approximation (solid), the solution for Maxwellian molecules (dash).

subseries of the original CE expansion. In other words, if we now expand the function $R_{MM}(g)$, Eq. (15), in powers of g, in the point $g=0$, we come back to the corresponding series inside the brackets in Eq. (8). That this is indeed true can be checked up to a few lower orders straightforwardly, though the complete proof requires a more involved analysis, and will be reported elsewhere. As the final comment to this point, we would like to stress a certain similarity between the problem considered above and the frequent situations in many-body problems: there is no single leading *term* but instead there is the leading *subseries* of the perturbation expansion, under certain conditions.

(ii) Let us discuss briefly the features of the resulting hydrodynamics. The hydrodynamic equations are now given by Eqs. (1a)–(1c), where σ is replaced with σ_{nl}, Eq. (9). First, the correction concerns the nonlinear regime, and thus the linearized form of new equations coincides with the linearized Navier-Stokes equations. Second, the solution (15) for the Maxwellian molecules and the result of approximation (16) for other models suggests that the modified viscosity μR gives a vanishing contribution in the limit of very high values of the average velocity. This feature seems to be of no surprise: if the average velocity is very high in comparison to other characteristic velocities (in our case, to the heat velocity), no mechanisms of momentum transfer are relevant except for the transfer with the stream. However, a cautious remark is in order since the original ''kinetic'' descriptions are Grad equations (1) and not the Boltzmann equation. In any case, the result can be used (at least) as a *derived* model of hydrodynamics in the highly nonequilibrium domain. On the other hand, the computations based on the invariance principle are possible also for the Boltzmann equation [4], and work in this direction is in progress.

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[6] Indeed, if U is a characteristic mean velocity, and $u=U\overline{u}$, where $\overline{u}$ is dimensionless, then the term $(\partial_x u)^{n+1}$ receives the factor U^{n+1} which is the highest possible order of U among the terms available in the nth order of the CE expansion.